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On approximate solutions of the Graetz problem with axial conduction

C. LAOHAKUL, C. Y. CHAN, K. Y. LOOK and C. W. TAN

The Cooper Union School of Engineering, The Cooper Union for the Advancement of Science and Art, Cooper Square, New York, NY 10003, U.S.A.

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Abstract—Two approximate solutions of the Graetz problem with axial conduction and a specified wall temperature (one for low Peclet numbers and the other for high Peclet numbers) are presented and compared favorably with the exact analytical solutions. With the proposed techniques, the approximate eigenvalues and eigenfunctions are obtainable explicitly and readily computable, unlike the methods for the exact results. Both low and high Peclet number approximate solutions give excellent agreements with the exact results when $Pe \leqslant 1$ and $Pe \geqslant 10$ respectively. In the limits, both approximate solutions will tend towards those exact ones. For the intermediate range of Peclet numbers between 1 and 10, either approximate technique gives surprisingly satisfactory results.

INTRODUCTION

THE PROBLEM of steady-state heat transfer, the Graetz problem, with axial conduction in laminar flow has been studied both analytically [1–9] and numerically [10–12] for various boundary conditions. A comprehensive literature review of the past efforts may be found in the papers of Papoutsakis, Ramkrishna and Lim [2, 3]. It is observed that in order to obtain exact solutions, the eigenvalues and eigenfunctions of the problem in each region have to be determined individually for each value of Peclet number, often requiring lengthy computations on a computer.

In this study, two approximate schemes, based on the finite integral transform technique proposed by Bayazitoglu and Ozisik [4, 5], are developed. It is noted that there is an arbitrariness in specifying the function used in the finite integral transform equation. In the work of Bayazitoglu and Ozisik [4, 5], this function was the parabolic velocity distribution of the flowing fluid. With this choice, it can be shown that, as Peclet number tends to infinity, their lowest order approximation will lead to the exact solution of the original Graetz-type problem without the axial conduction. Thus, it can be expected that the approximate solution under this scheme will improve as the Peclet number increases. On the other hand, when the Peclet number is small, a new scheme is needed. For small Peclet numbers, the axial conduction will predominate the forced convection making the velocity distribution less important in the heat transfer analysis; therefore, it is natural to choose the plug flow velocity distribution as the function used in the finite integral transform. With these two schemes, the entire range of Peclet numbers can be covered. It is further noted that the finite integral transform technique can be extended to include the analysis of multiple heating–cooling sections along the tube wall by using Green's function.

In this work, two approximate solutions are presented for a specified wall temperature, one for high Peclet number approximations and the other for low Peclet number approximations. The results obtained herein are compared favorably with the exact analytical solutions of Nagasue [1].

The advantage of the proposed approximate solutions is that the eigenvalues and eigenfunctions can be obtained explicitly from simple formulae, unlike those obtained by exact and numerical techniques. Moreover, the approximate solutions give excellent agreements with the exact results for the entire range of Peclet numbers.

PROBLEM DESCRIPTION

The problem that will be considered is that of steady-state heat transfer with axial conduction in laminar flow of Newtonian fluid in a circular tube. The wall temperature, T_w, is prescribed in a finite section between two semi-infinitely long inlet and outlet sections which are maintained at a uniform temperature, T_0. In terms of dimensionless variables, the governing equation is

$$(1-\xi^2)\frac{\partial \Theta}{\partial \eta} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\frac{(\xi \partial \Theta)}{\partial \xi} + \frac{1}{Pe^2}\frac{\partial^2 \Theta}{\partial \eta^2} \tag{1}$$

and its boundary conditions

$$\Theta(\xi, \pm\infty) = \frac{\partial \Theta}{\partial \xi}(0, \eta) = \Theta(1, \eta < 0) = \Theta(1, \eta > L) = 0$$

$$\Theta(1, L > \eta > 0) = 1.$$

The restriction that $\Theta(1, L > \eta > 0)$ is a step function can be easily removed by the use of Green's function where $\Theta(1, L > \eta > 0)$ is a function of η. Furthermore, if there is more than one heating section between the two semi-infinitely long tubes, the solution for this setup can likewise be constructed through the use of Green's function.

NOMENCLATURE

A_{ij}	matrix defined by integral (10) or integral (19)
B_i	diagonal elements of A_{ij}
C_i	constant defined by equation (23.2)
D_i	constant defined by equation (23.3)
E_i	constant defined by equation (25)
$F(\eta)$	arbitrary function of dimensionless wall temperature in the main heating section
${}_1F_1(a; b; x)$	Kummer function
$f(\xi)$	function used in the finite integral transform (2)
$J_0, J_1(x)$	Bessel functions
L	l/RPe, dimensionless length of the main heat transfer section
l	length of the main heat transfer section
N_i	constant defined by integral (5)
Pe	Peclet number, Ru_{max}/α
R	tube radius
r	radial distance
T	temperature
u_{max}	maximum fluid velocity
$Z_i(\xi)$	eigenfunctions
z	axial distance.

Greek symbols

α	thermal diffusivity
β_i	eigenvalues
Δ_i	constant defined by equation (14)
η	z/RPe, dimensionless axial distance
$\Theta(\xi, \eta)$	$(T-T_0)/(T_w-T_0)$, dimensionless temperature distribution
$\Theta_b(\eta)$	dimensionless bulk temperature
λ_i	roots of $J_0(x)$ or ${}_1F_1(1/2-x/4; 1; x)$
ξ	r/R, dimensionless radial distance
$\phi(\xi, \eta)$	dimensionless developing temperature distribution
$\Phi(\eta)$	the finite integral transform defined by equation (2).

ANALYSIS

Consider a finite integral transform of the form

$$\Phi_i(\eta) = \int_0^1 \xi f(\xi) Z_i(\xi) \phi(\xi, \eta)\, d\xi \quad (2)$$

and its inverse formula is given by

$$\phi(\xi, \eta) = \sum_{i=1}^{\infty} \Phi_i(\eta) Z_i(\xi)/N_i \quad (3)$$

where $Z_i(\xi)$ satisfies

$$(\xi Z_i')' + \lambda_i^2 \xi f(\xi) Z_i = 0$$
$$Z_i'(0) = Z_i(1) = 0. \quad (4)$$

Clearly, equation (4) satisfies the Sturm–Liouville condition. Hence, $Z_i(\xi)$ is an orthogonal function, namely

$$\int_0^1 \xi f(\xi) Z_i Z_j\, d\xi = \begin{cases} 0 & \text{if} \quad i \neq j \\ N_i & \text{if} \quad i = j \end{cases}. \quad (5)$$

Thus, the solutions of equation (1) for the three regions are given by

$$\Theta_1(\xi, \eta) = \phi_1(\xi, \eta) = \sum_{i=1}^{\infty} \Phi_{1i}(\eta) Z_i(\xi)/N_i \quad (\eta < 0) \quad (6)$$

$$\Theta_2(\xi, \eta) = 1 + \phi_2(\xi, \eta) = 1 + \sum_{i=1}^{\infty} \Phi_{2i}(\eta) Z_i(\xi)/N_i \quad (0 < \eta < L) \quad (7)$$

$$\Theta_3(\xi, \eta) = \phi_3(\xi, \eta) = \sum_{i=1}^{\infty} \Phi_{3i}(\eta) Z_i(\xi)/N_i \quad (n > L). \quad (8)$$

Low Peclet number approximation

For the low Peclet number approximation, consider $f(\xi) = 1$. $Z_i(\xi)$ is thus $J_0(\lambda_i \xi)$ and λ_i are the positive roots of $J_0(x)$ whose values are well tabulated in [13]. It can be shown that $\Phi_1(\eta)$ for each section must satisfy

$$\frac{1}{Pe^2} \frac{d^2\Phi_i}{d\eta^2} = \sum_{j=1}^{\infty} A_{ij} \frac{d\Phi_j}{d\eta} + \lambda_i^2 \Phi_i \quad (9)$$

where

$$A_{ij} = \frac{1}{N_j} \int_0^1 \xi(1-\xi^2) Z_i Z_j\, d\xi. \quad (10)$$

Clearly, equation (9) is an infinite set of coupled, second degree differential equations. Certainly, it is not practical to solve such a system of equations. An approximate solution can be achieved by recognizing that when the Peclet number is small, off-diagonal elements of A_{ij} contribute very little to equation (9). When the Peclet number is small, forced convection becomes less important. The parabolic velocity distribution plays a less important role and may well be replaced by the plug flow velocity. Thus, A_{ij} becomes

$$\frac{1}{N_i} \int_0^1 \xi Z_i Z_j\, d\xi$$

which is identically zero if $i \neq j$ by the orthogonality property of Z_i. Thus, the off-diagonal elements can be safely neglected when the Peclet number is small. With this in mind, equation (9) becomes

$$\frac{1}{Pe^2} \frac{d^2\Phi_i}{d\eta^2} = B_i \frac{d\Phi_i}{d\eta} + \lambda_i^2 \Phi_i \quad (11)$$

where

$$B_i = \frac{1}{N_i}\int_0^1 \xi(1-\xi^2)Z_i^2\,\mathrm{d}\xi = \frac{2}{3}\left(1+\frac{1}{\lambda_i^2}\right). \quad (12)$$

The solution of equation (11) for each section is straightforward. Constants associated with the solutions can be determined by equating the temperatures and axial flux at $\eta = 0$ and $\eta = L$ and taking the finite integral transform. The approximate solution of problem (1) can now be expressed explicitly as

$$\Theta_1(\xi,\eta) = \sum_{i=1}^{\infty} \frac{(1-B_i/\Delta_i)}{\lambda_i J_1(\lambda_i)}\{\exp(\beta_{-i}\eta) - \exp(\beta_{-i}(\eta-L))\}J_0(\lambda_i\xi) \quad (13.1)$$

$$\Theta_2(\xi,\eta) = 1-\sum_{i=1}^{\infty} \frac{J_0(\lambda_i\xi)}{\lambda_i J_1(\lambda_i)}\{(1+B_i/\Delta_i) \times \exp(\beta_{+i}\eta)+(1-B_i/\Delta_i)\exp(\beta_{-i}(\eta-L))\} \quad (13.2)$$

$$\Theta_3(\xi,\eta) = \sum_{i=1}^{\infty} \frac{(1+B_i/\Delta_i)}{\lambda_i J_1(\lambda_i)}\{\exp(\beta_{+i}(\eta-L)) - \exp(\beta_{+i}\eta)\}J_0(\lambda_i\xi) \quad (13.3)$$

where

$$\Delta_i = [B_i^2+4\lambda_i^2/Pe^2]^{1/2} \quad (14)$$

$$\beta_{\pm i} = (B_i \mp \Delta_i)/(2/Pe^2). \quad (15)$$

Another quantity of interest is the bulk temperature which is defined by [1] as

$$\Theta_b(\eta) = 4\int_0^1 \xi(1-\xi^2)\Theta(\xi,\eta)\,\mathrm{d}\xi. \quad (16)$$

After evaluating integral (16), the bulk temperature for the three sections of the tube can be written down explicitly as

$$\Theta_{b1}(\eta) = 16\sum_{i=1}^{\infty}\frac{(1-B_i/\Delta_i)}{\lambda_i^4} \times \{\exp(\beta_{-i}\eta)-\exp(\beta_{-i}(\eta-L))\} \quad (17.1)$$

$$\Theta_{b2}(\eta) = 1-16\sum_{i=1}^{\infty}\frac{1}{\lambda_i^4}\{(1+B_i/\Delta_i) \times \exp(\beta_{+i}\eta)+(1-B_i/\Delta_i)\exp(\beta_{-i}(\eta-L))\} \quad (17.2)$$

$$\Theta_{b3}(\eta) = 16\sum_{i=1}^{\infty}\frac{(1+B_i/\Delta_i)}{\lambda_i^4} \times \{\exp(\beta_{+i}(\eta-L))-\exp(\beta_{+i}\eta)\}. \quad (17.3)$$

High Peclet number approximation

For the high Peclet number approximation, consider $f(\xi) = 1-\xi^2$. Under this scheme,

$$Z_i(\xi) = \exp(-\lambda_i\xi^2/2)\,{}_1F_1(1/2-\lambda_i/4;1;\lambda_i\xi^2)$$

and λ_i are the roots of ${}_1F_1(1/2-\lambda_i/4;1;\lambda_i)$ and ${}_1F_1$ is the Kummer function. Indeed, Z_i and λ_i are the eigenfunctions and eigenvalues of the Graetz-type problem without the axial conduction. As before, instead of equation (9), one obtains

$$\frac{1}{Pe^2}\sum_{j=1}^{\infty} A_{ij}\frac{\mathrm{d}^2\Phi_j}{\mathrm{d}\eta^2} = \frac{\mathrm{d}\Phi_i}{\mathrm{d}\eta} + \lambda_i^2\Phi_i \quad (18)$$

where

$$A_{ij} = \frac{1}{N_j}\int_0^1 \xi Z_i Z_j\,\mathrm{d}\xi. \quad (19)$$

Once again, it is not practical to solve the system of equations (18). An approximate solution can be obtained by neglecting the off-diagonal elements of A_{ij}. According to the work of Bayazitoglu and Ozisik [4, 5], it is called the lowest order analysis. It is worth noting that as the Peclet number tends to infinity, the solution will tend to exact solution when axial conduction is omitted. Therefore, the approximate solution under this scheme is appropriate for high Peclet number consideration. Thus, equation (18) becomes

$$\frac{1}{Pe^2}B_i\frac{\mathrm{d}^2\Phi_i}{\mathrm{d}\eta^2} = \frac{\mathrm{d}\Phi_i}{\mathrm{d}\eta} + \lambda_i^2\Phi_i \quad (20)$$

and B_i is the diagonal elements of A_{ij}.

In the same way as the low Peclet number approximation, one can derive the following expressions for the high Peclet number approximation:

$$\Theta_1(\xi,\eta) = \sum_{i=1}^{\infty}\frac{\beta_{+i}C_iZ_i}{(\beta_{-i}-\beta_{+i})} \times \{\exp(\beta_{-i}(\eta-L))-\exp(\beta_{-i}\eta)\} \quad (21.1)$$

$$\Theta_2(\xi,\eta) = 1+\sum_{i=1}^{\infty}\frac{C_iZ_i}{(\beta_{-i}-\beta_{+i})} \times \{\beta_{+i}\exp(\beta_{-i}(\eta-L))-\beta_{-i}\exp(\beta_{+i}\eta)\} \quad (21.2)$$

$$\Theta_3(\xi,\eta) = \sum_{i=1}^{\infty}\frac{\beta_{-i}C_iZ_i}{(\beta_{-i}-\beta_{+i})} \times \{\exp(\beta_{+i}(\eta-L))-\exp(\beta_{+i}\eta)\} \quad (21.3)$$

and the bulk temperatures for the three regions are

$$\Theta_{b1}(\eta) = 4\sum_{i=1}^{\infty}\frac{\beta_{+i}C_iD_i}{(\beta_{-i}-\beta_{+i})} \times \{\exp(\beta_{-i}(\eta-L))-\exp(\beta_{-i}\eta)\} \quad (22.1)$$

$$\Theta_{b2}(\eta) = 1+4\sum_{i=1}^{\infty}\frac{C_iD_i}{(\beta_{-i}-\beta_{+i})} \times \{\beta_{+i}\exp(\beta_{-i}(\eta-L))-\beta_{-i}\exp(\beta_{+i}\eta)\} \quad (22.2)$$

$$\Theta_{b3}(\eta) = 4\sum_{i=1}^{\infty}\frac{\beta_{-i}C_iD_i}{(\beta_{-i}-\beta_{+i})} \times \{\exp(\beta_{+i}(\eta-L))-\exp(\beta_{+i}\eta)\} \quad (22.3)$$

where

$$B_i = C_iZ_i'(1)/2+2 \quad (23.1)$$

$$C_i = D_i/N_i \quad (23.2)$$

$$D_i = -Z_i'(1)/\lambda_i^2 \quad (23.3)$$

$$N_i = \int_0^1 \xi(1-\xi^2)Z_i^2 \, d\xi \qquad (23.4)$$

$$Z_i'(1) = \lambda_i(1-\lambda_i/2)\exp(-\lambda_i/2) \times {}_1F_1(3/2-\lambda_i/4;2;\lambda_i) \qquad (23.5)$$

$$\beta_{\pm i} = (1 \mp [1+4B_i\lambda_i^2/Pe^2]^{1/2})/(2B_i/Pe^2). \qquad (23.6)$$

It should be pointed out that it is not necessary to evaluate the integral N_i since the values of C_i are well tabulated in any work related to the original Graetz-type problem [14, 15].

Approximate solutions by Green's function

In previous sections, two approximate solutions were constructed for the case of step change in temperature in the heating section of the tube (i.e. $0 < \eta < L$). Through the use of Green's function, one can easily derive a solution for temperature distribution for the case where the wall temperature is an arbitrary function of axial distance, $F(\eta)$. Similar to the work of [1, 16], the approximate solutions based on Green's function are:

$$\Theta_1(\xi,\eta) = \sum_{i=1}^{\infty} E_i Z_i \times \left\{ \int_0^L \exp(\beta_{-i}(\eta-\eta'))F(\eta')\,d\eta' \right\} \qquad (24.1)$$

$$\Theta_2(\xi,\eta) = \sum_{i=1}^{\infty} E_i Z_i \left\{ \int_0^\eta \exp(\beta_{+i}(\eta-\eta'))F(\eta')\,d\eta' + \int_\eta^L \exp(\beta_{-i}(\eta-\eta'))F(\eta')\,d\eta' \right\} \qquad (24.2)$$

$$\Theta_3(\xi,\eta) = \sum_{i=1}^{\infty} E_i Z_i \times \left\{ \int_0^L \exp(\beta_{+i}(\eta-\eta'))F(\eta')\,d\eta' \right\} \qquad (24.3)$$

where

$$E_i = \beta_{-i}\beta_{+i}Z_i'(1)/[N_i(\beta_{-i}-\beta_{+i})\lambda_i^2]. \qquad (25)$$

Of course, $Z_i(\xi)$, $\beta_{\pm i}$, N_i and λ_i must be used appropriately depending on the types of approximations being considered.

Indeed, Green's function not only allows the wall temperature in the heating section to vary arbitrarily, but it also provides a convenient way of constructing the solutions for the case in which the heating section can be divided into a series of heating and cooling subsections of arbitrary lengths, provided that the total length is L.

RESULTS AND CONCLUSION

Figure 1 shows the bulk temperatures for $Pe = 1$ and $L = 1$, and $Pe = 10$ and $L = 0.1$ for a unit step change in temperature in the heating section of the tube. It is evident that the low Peclet number approximate scheme gives a more accurate comparison when

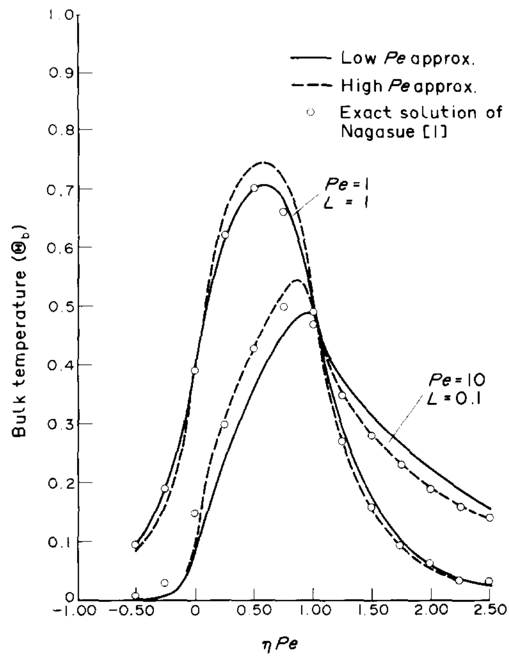

FIG. 1. Bulk temperature profiles.

compared with the exact results of Nagasue [1] for $Pe = 1$ and $L = 1$. On the other hand, the high Peclet number approximate scheme gives a more accurate comparison for $Pe = 10$ and $L = 0.1$. This observation indeed confirms the theory developed in the previous sections regarding the appropriateness of the two proposed approximate schemes. It is also apparent that either approximate technique gives surprisingly good comparison at either $Pe = 1$ or $Pe = 10$.

Thus, it can be concluded that when $Pe \leqslant 1$, the low Peclet number approximate scheme should be used; when $Pe \geqslant 10$, the high Peclet number approximate scheme is recommended; and when $10 > Pe > 1$, either method is appropriate.

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SOLUTIONS APPROCHEES DU PROBLEME DE GRAETZ AVEC CONDUCTION AXIALE

Résumé—Deux solutions approchées du problème de Graetz avec conduction axiale et température pariétale donnée (une pour les nombres de Péclet faibles et l'autre pour les grands nombres de Péclet) sont présentées et favorablement comparées aux solutions analytiques exactes. Avec les techniques proposées, les valeurs propres approchées et les fonctions propres sont explicitées et calculables aisément, contrairement aux méthodes qui donnent les résultats exacts. Les deux solutions approchées donnent un accord excellent avec les résultats exacts lorsque $Pe \leqslant 1$ et $Re \geqslant 10$ respectivement. Aux limites, les deux solutions approchées tendent vers les solutions exactes. Pour le domaine intermédiaires des nombres de Péclet, entre 1 et 10, l'une ou l'autre technique approachée donne de façon surprenante des résultats satisfaisants.

NÄHERUNGSLÖSUNGEN DES GRAETZ-PROBLEMS MIT AXIALER WÄRMELEITUNG

Zusammenfassung—Zwei Näherungslösungen des Graetz-Problems mit axialer Wärmeleitung und definierter Wandtemperatur (eine Lösung für niedrige Peclet-Zahlen und eine für hohe Peclet-Zahlen) werden vorgestellt und mit den exakten analytischen Lösungen verglichen. Bei dieser Vorgehensweise werden die angenäherten Eigenwerte und Eigenfunktionen explizit ermittelt. Sie sind im Gegensatz zu den Methoden bei der exakten Lösung leicht zu berechnen. Sowohl bei niedrigen wie auch bei hohen Peclet-Zahlen liefern die Näherungslösungen ausgezeichnete Übereinstimmungen mit den exakten Ergebnissen für $Pe \leqslant 1$ bzw. $Pe \geqslant 10$. Als Grenzwerte der beiden Näherungslösungen ergeben sich die exakten Lösungen. Für den Übergangsbereich ($1 < Pe < 10$) erhält man mit beiden Näherungsmethoden überraschend gute Ergebnisse.

ПРИБЛИЖЕННОЕ РЕШЕНИЕ ЗАДАЧИ ГРЕТЦА С АКСИАЛЬНОЙ ПРОВОДИМОСТЬЮ

Аннотация—Представлены два приближенных решения задачи Гретца с учетом аксиальной проводимости при заданной температуре стенки (одно для малых, другое для больших чисел Пекле). Решения сравниваются с точными аналитическими. С помощью этого метода можно получить в явном виде и легко рассчитать приближенные собственные значения и функции в отличие от методов для получения точных решений. Приближенные решения и с малыми, и с большими числами Пекле хорошо согласуются с точными при $Pe \leq 1$ и $Pe \leq 10$, соответственно. В этих пределах оба приближенных решения стремятся к аналогичным точным. В промежуточном диапазоне чисел Пекле от 1 до 10 любой приближенный метод дает вполне удовлетворительные результаты.